Answers and Solutions

1. **Answer**: 200.
   Solution: We use the following identity:

   $$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ac)$$

2. **Answer**: 243.
   Solution: If we raise the given matrix to the n-th power, the result will be a matrix, all elements of which equal 3^{n-1} . This can easily be proven using induction.

3. **Answer**: ½.
   Solution: Let us examine the axial sections of the goblet and the ball. We need to find the maximal radius of a circle, which has only one common point with the parabola $y=x^2$ – at the origin. Since the circle is tangent to the X axis, its equation will have the following form: $x^2 + (y - r)^2 = r^2$, where r is the radius. Let us transform this equation into: $x^2 = 2ry - y^2$. For the points of the circle, at $x\neq 0$, the condition $y > x^2$ must be satisfied . Therefore $y>2ry - y^2$, or $y(y-(2r-1))>0$. The last condition must be satisfied for any $y>0$, which means that $r \leq 1/2$.

4. **Answer**: 59.
   Solution: Dividing a number and the sum of its digits by 9, we obtain the same remainder, therefore, the sum of the two numbers written on each road sign (which is equal to the length of the road), when divided by 9, will have a remainder of 5.
   If the length of the road n is greater than 59, the sum of all the digits of the numbers on the road sign (59; n-59) is greater than 14.
   In addition, the length of the road cannot equal any of the following numbers: 5, 14, 23, 32, 41 and 50, since in each of these cases the number of digits on at least one of the road signs would equal 5. These road signs are (1:4), (1:13), (1:22), (1:31), (1:40) and (10:40), respectively.
   Let us prove that if the length of the road equals 59 km, the conditions of the problem will be satisfied. If the number on one side of the road sign is $\overline{ab} = 10a + b$ (the line above a and b signifies that they are the digits of a two-digit number), then the number on the other side of the sign must be $\overline{(5 - a)(9 - b)} = 10(5 - a) + (9 - b)$. Therefore, the sum of digits of this sign equals: $a + b + (5 - a) + (9 - b) = 14$.

5. **Answer**: 1024.
   Solution: The function maps the unit interval [1,0] in such a way that both of

its ends are mapped into 0, and its middle is mapped into 1. On each of the intervals – [0, 0.5] and [0.5, 1] – the function is linear. Therefore, the behavior of the tenfold iteration of this function can be described as follows:
if we divide the unit interval into 2^{10} equal parts, then on each of these parts of the unit interval the tenfold iteration of the function will be linear. On the odd-numbered parts the tenfold iteration of the function will increase from 0 to 1, while on even-numbered parts it will decrease from 1 to 0. (The graph will resemble a toothed saw). Therefore, each of these segments intersects the line $y = x$ exactly once. The equation has no solutions outside the unit segment.

6. **Answer**: 1/10.
   Solution: Let us denote the integral as *I*, and substitute *x* for *π/2-x*. We have

$$I = \int_0^{\pi/2} (x^4 + (x-\pi/2)^4)$$

which means that,

$$I = \frac{1}{2}\int_0^{\pi/2} (x^4 + (x-\pi/2)^4)(\sin^2 x + \cos^2 x)dx =$$
$$= \frac{1}{2}\int_0^{\pi/2} (x^4 + (x-\pi/2)^4)dx = \frac{1}{2}\cdot 2\int_0^{\pi/2} x^4 dx = \frac{x^5}{5}\Big|_0^{\pi/2} = \frac{\pi^5}{160}$$

Therefore,

$$\frac{16}{\pi^5} I = \frac{1}{10}$$

7. **Answer**: 84
   Solution: First, let us calculate the maximal number of different pairs of cube faces, which can be glued together. If the two faces are colored differently, then the number of pairs is $C_6^2 = 15$, and if they are of the same color, then the number of pairs is 6. In each of these 21 cases, we can glue the cubes together in 4 different ways (by turning one of the cubes at 90 degree angles). Therefore, there are $21\cdot 4=84$ different ways of gluing the cubes together.

8. **Answer**: 4.
   Solution: Let us examine the prime factorization of the number:

$$2010! = 2^{n_2}\cdot 3^{n_3}\cdot 5^{n_5}\cdot ...$$

   In order to solve the problem we need to find the last digit of the number $N = \frac{2010!}{10^{n_5}}$. It is obvious that $n_2 > n_5$, which means that the digit that we're looking for is even. And therefore, all we have to do is find its remainder when divided by 5. Let us denote the product of all integers from 1 to *n*, excluding those divisible by 5, as *p(n)* . By multiplying separately first, numbers that are not divisible by 5, then those that are divisible by 5 but not by 25, then those that are divisible by 25 but not by 125, etc., we obtain

$$N = \frac{p(2010)p(402)p(80)p(16)p(3)}{2^{n_5}}$$

And in addition, $n_5 = 402 + 80 + 16 + 3 = 501$. Let us note that $1 \cdot 2 \cdot 3 \cdot 4 \equiv -1 (mod\ 5)$
Therefore, $p(2010) \equiv (-1)^4 20 \equiv 1 (mod\ 5)$.
Similarly, $p(420) \equiv 2 (mod\ 5)$, $p(80) \equiv 1 (mod\ 5)$, $p(16) \equiv 4 (mod\ 5)$, $p(3) \equiv 1 (mod\ 5)$.
In addition, $2^{501} \equiv 2(\text{mod } 5)$.
This means that

$$N \equiv \frac{1 \cdot 2 \cdot 1 \cdot 4 \cdot 1}{2} \equiv 4 (\text{ mod } 5)$$

Thus, the digit is even, and when divided by 5, has a remainder of 4. Therefore, the digit is 4.

9. **Answer**: 6.

Solution: First, let us note that the projection of the polyline on each coordinate axis has the length of at least $2\sqrt{2}$ - twice the length of the cube's edge. Let x_y denote the sum of projections of the polyline segments belonging to faces parallel to the xy plane on the X axis. Let y_x denote the sum of projections of the same segments on the *Y* axis. The notations x_z, z_x, y_z and z_y are introduced similarly. The length of the polyline is no less than

$$\sqrt{x_y^2 + y_x^2} + \sqrt{x_z^2 + z_x^2} + \sqrt{z_y^2 + y_z^2}$$

Utilizing the mean square and arithmetic mean inequality, we obtain

$$\frac{\sqrt{2}}{2}((x_y + y_x) + (x_z + z_x) + (z_y + y_z)) = \frac{\sqrt{2}}{2}((x_y + x_z) + (y_x + y_z) + (z_x + z_y)) \geq$$
$$\geq \frac{\sqrt{2}}{2} \cdot 2\sqrt{2} \cdot 3 = 6$$

The length equals 6 for a polyline which is perpendicular to a main diagonal of the cube (and intersects all of its faces). For instance, if the plane which the polyline belongs to contains the center of the cube, then the polyline is a regular hexagon, the length of each edge of which equals 1.