Internet Olympiad (First Round)

1. On Mondays, Wednesdays and Fridays one dollar is converted to 31 roubles, on Tuesdays, Thursdays and Saturdays one dollar is converted to 30 roubles. Conversions are not executed on Sundays. Starting from the initial amount of 1000 dollars how many weeks will it take to increase this amount to 2700 dollars taking advantage of the fluctuations in exchange rates ?

   Answer: 11 weeks.

   Solution: In two workdays the initial amount of dollars, x, can be increased by $\frac{1}{30}$, i.e. it can be turned into (1+1/30)x by converting the dollars into roubles at a rate of 1 dollar for 31 roubles and then converting the roubles into dollars at a rate of 30 roubles for a dollar the following day. There are three pairs of workdays in a week. After 31 pairs of workdays the following sum of dollars can be obtained: $2700\$ < 1000e\$ < 1000 \cdot (1+\frac{1}{30})^{31}\$$, while $(1+\frac{1}{30})^{30} < 2700$, i.e. another 9 pairs of workdays are necessary.

2. Find $\lim_{x\to\infty}\left(1 - x^{\frac{1}{2x}}\right)\cdot\frac{x}{2\ln x}$.

   Express the answer as a decimal fraction.

   Answer: $-1/4$.

   Solution:

$$\lim_{x\to\infty}\left(1 - x^{\frac{1}{2x}}\right)\cdot\frac{x}{2\ln x} = \lim_{x\to\infty}\left(1 - e^{\frac{\ln(x)}{2x}}\right)\cdot\frac{2x}{\ln x}\cdot\frac{1}{4}.$$

   Suppose that $t = \frac{\ln(x)}{2x}$; $t \to 0$ for $x \to \infty$. Then we have

$$\lim_{x\to\infty}\left(1 - e^{\frac{\ln(x)}{2x}}\right)\cdot\frac{2x}{\ln x}\cdot\frac{1}{4} = \lim_{t\to 0}\left(1 - e^{t}\right)/t\cdot\frac{1}{4} = -\frac{1}{4}.$$

   .

3. Find the maximum number of parts into which a plane can be divided by the graphs of 10 quadratic trinomials $a_ix^2 + b_ix + c_i$, $i = 1, \dots, 10$.

   Answer: 101.

   Solution:

   Any two graphs of quadratic trinomials can intersect at no more than two points. If a trinomial intersects with each of k other trinomials at two points, then the plane is divided into $2k + 1$ parts. The first trinomial divides the plane into $2 = 1 + 1$ parts. Thus, if we have k graphs of quadratic trinomials, each pair of which intersect at two points, then the number of parts the plane will be divided into equals $1+1+3+\ldots+2k-1=k^2+1$. If k=10, then $k^2+1=101$.

Now let us build an example where any two graphs intersect at exactly two points (and all these points are different for all the pairs of trinomials). In order to do that, we can examine tapering graphs of trinomials with vertex at zero and decreasing ordinate of vertex. In other words, let a_i be positive increasing numbers, $b_i=0$ for any i, and c_i be decreasing numbers.

4. Calculate $\int_{-5}^{5}\frac{dx}{1+2^{\arctan x}}$.

   Answer: 5.

   Solution:

   Since the function $arctg(x)$ is symmetrical and odd, we obtain

$$\int_{-5}^{5}\frac{dx}{1+2^{\arctan x}}=\int_{-5}^{5}\frac{dx}{1+2^{-\arctan x}}$$

.

   Hence,

$$\int_{-5}^{5}\frac{dx}{1+2^{\arctan x}}=\frac{\int_{-5}^{5}\frac{dx}{1+2^{\arctan x}}+\int_{-5}^{5}\frac{dx}{1+2^{-\arctan x}}}{2}=$$

$$=\int_{-5}^{5}\frac{dx}{2}\cdot\left(\frac{1}{1+2^{\arctan x}}+\frac{1}{1+2^{-\arctan x}}\right)=\int_{-5}^{5}\frac{dx}{2}\cdot\left(\frac{1}{1+2^{\arctan x}}+\frac{2^{\arctan x}}{1+2^{\arctan x}}\right)=$$

$$=\int_{-5}^{5}\frac{dx}{2}\cdot\left(\frac{1+2^{\arctan x}}{1+2^{\arctan x}}\right)=\int_{-5}^{5}\frac{dx}{2}\cdot 1=5.$$

5. The sequence $\{a_n\}_{n=1}^{\infty}$, $a_n=\frac{n}{\pi}\sin(2\pi en!)$ is given. Find $\lim_{n\to\infty}a_n$.

   Answer: 2.

   Solution: $e=\sum_{k=0}^{\infty}\frac{1}{k!}$. Consequently,

$$en!=\sum_{k=0}^{n}\frac{n!}{k!}+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots=$$

$$=K+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots$$

   for some integer K. Therefore,

$$\sin(2\pi en!)=\sin 2\pi\left(\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots\right),$$

which means that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\pi}\cdot 2\pi \left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots\right),$$

since $\sin(t)/t \to 1$ for small t, and hence the sinus can be removed. In our case,

$$t = 2\pi \left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots\right).$$

Next, we obtain $\lim\limits_{n\to\infty} \frac{n}{\pi}\cdot 2\pi \left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots\right) =$

$$= \lim_{n\to\infty} 2n \cdot \frac{1}{n+1} = 2.$$

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6. $e^A = \sum_{n=0}^{\infty} A^n/n!$ $(A^0 = E)$. It is known that $AB \neq BA$, and the coefficients A and B are real. Is it possible that $e^A e^B = e^B e^A$? If the answer is positive, give an example.

   Answer: It is possible.

   Solution: Let $A = \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}$, and B be an arbitrary matrix that does not commute with A. It is sufficient to demonstrate that $e^A = -E$. If so, the matrix commutes with all matrices, including e^B. Let $B = \begin{pmatrix} 0 & \varphi \\ -\varpi & 0 \end{pmatrix}$. Then $B^{2k} = \begin{pmatrix} (-1)^k\varphi^{2k} & 0 \\ 0 & (-1)^k\varphi^{2k} \end{pmatrix}$ and

$$B^{2k-1} = \begin{pmatrix} 0 & (-1)^k\varphi^{2k-1} \\ (-1)^{k+1}\varphi^{2k-1} & 0 \end{pmatrix}.$$

   Therefore,

$$e^B = \begin{pmatrix} \sum_{k=0}^{\infty}(-1)^k \frac{\varphi^{2k}}{(2k)!} & \sum_{k=0}^{\infty}(-1)^k \frac{\varphi^{2k+1}}{(2k+1)!} \\ -\sum_{k=0}^{\infty}(-1)^k \frac{\varphi^{2k+1}}{(2k+1)!} & \sum_{k=0}^{\infty}(-1)^k \frac{\varphi^{2k}}{(2k)!} \end{pmatrix} = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix}.$$

   In our case, $\varphi = \pi$, and therefore, $A = -E$.

7. The sequence $Q_1(x) = x, Q_{n+1}(x) = \frac{Q_n(x+1)}{Q_n(x)}$, $n \geq 1$ is given. Let $Q_n(x) - 1 = A_n(x)/B_n(x)$, where A and B are polynomials. Find the ratio of the leading coefficients of $A_7(x)$ and $B_7(x)$.

   Answer: -5!=-120.

   Solution:

   Let us denote by c_n the ratio of the leading coefficients $A_n(x)$ and $B_n(x)$. Obviously, for some integer $k \geq 0$, $Q_n(x) = 1 + \frac{c_n}{x^k} + o(\frac{1}{x^k})$

as $x \to \infty$. Consequently, $\ln(Q_n(x)) = \frac{c_n}{x^k} + o(\frac{1}{x^k})$ as $x \to \infty$ since $\ln(1+t)/t \to 1$, at $t \to 0$ (in our case $t = \frac{c_n}{x^k} + o(\frac{1}{x^k})$). Let us denote by $\Delta(f) = f(x+1) - f(x)$ the difference derivative operator with step size 1. Then $\Delta(\ln(x)) = \frac{1}{x} + \sum_{k=2}^{\infty} d_k \frac{1}{x^k}$ and

$$\Delta\left(\frac{1}{x^k}\right) = -k\frac{1}{x^{k+1}} + \sum_{l=k+1}^{\infty} c_l \frac{1}{x^{k+l}}.$$

(The asymptotic behavior of the difference derivative is the same as the asymptotic behavior of the regular derivative). It follows that $\Delta^6(\ln(x))$ has the same leading coefficient as $\ln(x)^{(6)}$, i.e. $(-1)^5 5!$.

8. Let V_T designate the volume of the unit neighborhood of a regular tetrahedron T with unit edge (i.e. the set of points for which the distance from the tetrahedron is less or equal to one). Let V_O designate the volume of the unit neighborhood of a regular octahedron O with unit edge. (Regular octahedron is an octahedron with 8 equilateral triangles as faces). Find $V_O + 2V_T$.

   Answer: $\frac{\sqrt{2}}{2} + 4\sqrt{3} + 16\pi$.

   Solution: Let $V_M(\varepsilon)$ denote the volume of the -neighborhood of the convex polyhedron M.

   $$V_M(\varepsilon) = c_0 + c_1\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3,$$

   where c_0 is the volume of M, c_1 is its surface area, $c_2 = \sum l_i(\pi - \alpha_i)$ (l_i is the length of its $i-th$ edge, α_i is the corresponding dihedral angle, $c_3 = 4\pi/3$ is the volume of a unit sphere).

   The volume of a regular tetrahedron with edge $\sqrt{2}$ is $1/3$. (It can be inscribed in a unit cube, so that the remaining part of the cube consists of 4 tetrahedrons the volume of each of which is $1/6$). It follows that the volume of a regular tetrahedron with edge 1 is $\frac{1}{3\sqrt{8}}$. An octahedron with edge $\sqrt{2}/2$ can be inscribed in a unit cube connecting the centers of its faces. This octahedron is the union of two pyramids the with base area of which equals $1/2$, and the altitude $1/2$. Therefore, its volume equals $1/6$. It follows that the volume of a unit octahedron equals $\frac{\sqrt{2}}{3}$, and if we add twice the volume of a unit tetrahedron the sum will be $\frac{\sqrt{2}}{2}$. The area of an equilateral unit triangle equals $\sqrt{3}/4$, four such triangles form a tetrahedron, and eight - an octahedron. Hence, their surface areas equal $\sqrt{3}$ and $2\sqrt{3}$, respectively, and their sum equals $4\sqrt{3}$.

   The dihedral angle of a tetrahedron together with the dihedral angle of an octahedron equal π, since the continuation of the faces of a regular octahedron inscribed in a cube is the union of two regular tetrahedra inscribed into it. A tetrahedron has 6 edges, and an octahedron - 12. Therefore, the sum of the coefficients of ε^2 will equal 12π. Summing up the components and assuming that $\varepsilon = 1$, we obtain the answer.