Individual Olympiad

1. Find the antiderivative $\int \ln(x+\sqrt{x^2+1})dx$.

Solution:

$$\int \ln(x+\sqrt{x^2+1})dx = x\ln(x+\sqrt{x^2+1}) - \int x d\ln(x+\sqrt{x^2+1}) = x\ln(x+\sqrt{x^2+1}) - \int \frac{x}{\sqrt{x^2+1}}dx =$$

$$x\ln(x+\sqrt{x^2+1}) - \sqrt{x^2+1}$$

2. Find $\lim_{x\to 0}\frac{sin(x)-\arcsin(x)}{x^3}$.

Solution: $\sin(x) = x - \frac{x^3}{6} + o(x^3)$, and therefore $\arcsin(x) = x + \frac{x^3}{6} + o(x^3)$. Hence, $sin(x) - \arcsin(x) = -\frac{x^3}{3} + o(x^3)$. Consequently, $\lim_{x\to 0}\frac{sin(x)-\arcsin(x)}{x^3} = -\frac{1}{3}$.

3. At the initial moment the number 0 is written on a board. Every second the number that was written previously, x, is erased, and the numbers $x-1$ and $x+1$ are written instead of it, so that by the 10th second 1024 numbers will be written. Find the average of their squares.

Solution: Note that $\frac{(x-1)^2+(x+1)^2}{2} = x^2+1$. Hence $\frac{\sum_{i=1}^k (x_i+1)^2+\sum_{i=1}^k (x_i-1)^2}{2k} = \frac{\sum_{i=1}^k x_i^2}{k}+1$. Thus, the operation increases the arithmetic mean of the squares of the recorded numbers by 1. At first it was zero, and after 10 operations the result will be 10.

4. Regarding the matrices A, B, C it is known that $AB = BA, CB = BC$, and that $B \neq \lambda E$ for any λ (E is the identity matrix, λ is a complex number). Is it true that $AC = CA$?

Answer: Not necessarily.

Solution. Let V_1 and V_2 be subspaces of the four - dimensional matrix space V, $V = V_1 + V_2$, $dim(V_1) = dim(V_2) = 2$, $BV_1 = V_1$, $AV_2 = V_2$, $CV_2 = V_2$, projections of A and C on V_1 do not commute.

Now let's give a counter-example in explicit form.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

5. Baron Munchausen has a sufficient number of boards - segments of length 1 meter. He wants to get to the center of a lake the radius of which is 1 kilometer. He places the boards one after another. Each end of each board can be placed either on the shore or on a previously placed board. Can he do it?

Answer: Yes, he can!

Solution. If you put the ends of a board on a circle of radius R, then its middle will be situated at a distance ΔR from its border. And in this case ΔR increases with decreasing R. The boards can easily be placed in such a way that a rim the width of $\Delta R/2$ would be separated from the rest of the lake by a board. Repeating the process, we get to the center of the lake.

6. Is it possible to position on a plane 32 circles in such a way that each of them touches either 5 or 6 others?

Answer: It is possible. Consider a soccer ball - a polyhedron with 12 regular pentagonal and 20 hexagonal faces. Into each of them we inscribe a circle. Next, we stereographically map the sphere into a plane. The circles will be moved into circles.

7. Prove that all complex roots of polynomial $P(x) = 1 + x + x^2/2 + \ldots x^n/n!$ are modulo at least 1.

Solution. Clearly, it suffices to consider the case $x < 0$. $P(x)$ is a partial sum of the Taylor series for e^x. According to the theorem on the remainder $P(x) - e^x = \theta^{n+1}/(n+1)!$ for some $\theta \in [x, 0]$. If $|x| < 1$, then $|\theta| < 1$. If in this case $P(x) = 0$, then $\theta^{n+1}/(n+1)! = -e^x$. Hence $|\theta^{n+1}/(n+1)!| = |e^{-x}| > e^{-1}$. In addition, $1/(n+1)! > |\theta^{n+1}/(n+1)!|$ since $|\theta| < |x| < 1$. Thus we have $1/(n+1)! > e^{-1}$, which for $n \geq 2$ is impossible. The case where $n = 1$ is to be considered separately.

8. n points X_i are marked on a unit circle. Prove that it is possible to add another point, Y, such that the product of the distances $|YX_i|$ would be greater than 1.

Solution. Consider the circle as a unit circle on a complex plane. Consider the polynomial $\prod(x - x_i)$. The module of its value at the origin is 1. The module of its value at point X of the unit circle is equal to $\prod |x - x_i|$. It is known that the maximum absolute value of an analytic function (such as a polynomial) is achieved on the boundary. Consequently, the value $\prod |x - x_i|$ (i.e. the product of the distances to the marked points) at some point of the unit circle is definitely greater than unity.

9. There is an unlimited number of boxes, 10 chips can be placed in each box. The first player puts two white chips in any two boxes or both chips in one of the boxes, the second player puts one black chip in any box.

The goal of the first player is to fill one of the boxes with 10 white chips.

a) Can the first player achieve this goal?

b) What is the minimal number of moves needed for this?

Answer a) Yes. **b)** 128.

Solution. a)

Let the first player begin with putting one chip in two different boxes during every move. The second player can only put a black chip in one box, so that after every move we get one additional box with one white chip. Thus, it is possible to obtain n boxes with one white chip in each. Now, let the first player choose two of these boxes during every move and add a chip to each. The second player can only 'spoil' one box by putting a black chip in it, and consequently we obtain $n/2$ boxes exactly with two white chips in them. The first player can continue this process, and eventually obtain $n/1024$ boxes filled with white chips.

b) Let the *weight* of a box be defined as the value $2^n - 1$, if the box contains exactly n white chips and nothing else (in particular, $2^0 - 1 = 0$, if the box is empty), and if the box contains a black chip - then let its *weight* be 0. Let black put his chip in the box which contains no black chips and the maximal number of white chips. In this case the total weight of the boxes after white makes his move and black responds will increase by no more than 1. During the last move white fills a box with two chips after which the game is terminated (and so are black's moves). The smallest possible total weight in this situation is 127, and another one 128- th and final move.