Question 1

In a study of the effectiveness of a new modification of a certain medication the participants were divided into two groups: the experimental group, in which the patients were given the new version of the medication, and the control group, in which the patients were given the old medication. Each group had the same number of people. The percentage of women and the percentage of men who recovered within a week after starting the medication was calculated separately for each group. It is known that for both women and men, the percentage of those who recovered within a week after starting the medication was higher in the experimental group than in the control group.

Can we conclude from this that the number of people in the experimental group who recovered within a week after starting the medication was higher in the experimental group than in the control group?

Answer: No, we cannot.

Solution: Let us examine the following example.

Suppose that there were 1000 people in each group. In the experimental group there were 300 women, 70% of whom recovered within a week, and 700 men, 40% of whom recovered within a week, and in the control group there were 700 women, 60% of whom recovered within a week, and 300 men, 30% of whom recovered within a week.

Thus, in the experimental group $300 \cdot 0{,}7 + 700 \cdot 0{,}4 = 490$ people recovered within a week, and in the control group – $700 \cdot 0{,}6 + 300 \cdot 0{,}3 = 510$ people recovered within a week.

Question 2

Find the indefinite integral

$$\int \frac{1}{x \cdot \ln x \cdot \ln\ln x \cdot \ln\ln\ln x \cdot \ln\ln\ln\ln x \cdot \ln\ln\ln\ln\ln x} dx .$$

Answer. $\ln\ln\ln\ln\ln\ln x + C$.

Solution.

$$\int \frac{1}{x \cdot \ln x \cdot \ln\ln x \cdot \ln\ln\ln x \cdot \ln\ln\ln\ln x \cdot \ln\ln\ln\ln\ln x} dx =$$

$$\int d\left(\ln\ln\ln\ln\ln\ln x\right) = \ln\ln\ln\ln\ln\ln x + C$$

Question 3

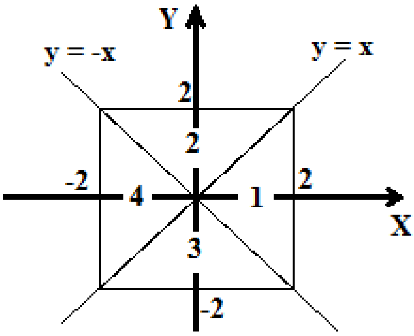
a) Find the area defined by the inequality: $|x+y| + |x-y| \le 4$.

Answer. 16

Solution.

We obtain four systems which correspond to four areas on a plane (see the figure below), which together form a square the area of which is 16.

$$1)\begin{cases} x \ge -y \\ x \ge y \\ x \le 2 \end{cases} \quad 2)\begin{cases} x \ge -y \\ x < y \\ y \le 2 \end{cases} \quad 3)\begin{cases} x < -y \\ x \ge y \\ y \ge -2 \end{cases} \quad 4)\begin{cases} x < -y \\ x < y \\ x \ge -2 \end{cases} .$$



b) Find the volume defined by the inequality:
$|x+y+z|+|x+y-z|+|x-y+z|+|-x+y+z| \le 4$.

Answer. $20/3$.

Solution. Let $F(x;y;z)=|x+y+z|+|x+y-z|+|x-y+z|+|-x+y+z|$. Note that $F(x;y;z)=F(x;y;-z)=F(x;-y;z)=F(-x;y;z)$, and therefore, this region is symmetric about all coordinate planes. Let us find the volume of the part of this region which is located in the 1-st octant. Let x be the greatest of x, y, z. Then $x+y-z \ge 0$ and $x-y+z \ge 0$. If $-x+y+z \ge 0$, then the following inequality holds true $(x+y+z)+(x+y-z)+(x-y+z)+(-x+y+z)=2(x+y+z)\le 4$, и $(x+y+z)\le 2$. If $-x+y+z \le 0$, then $(x+y+z)+(x+y-z)+(x-y+z)+(x-y-z)=4x \le 4$, $x \le 1$, and consequently $y \le 1, z \le 1$. Thus, the part of the figure which is located in the 1-st octant is part of the cube $0 \le x, y, z \le 1$, which is bounded by the plane $x+y+z=2$. Since the cube vertices (0, 1, 1), (1, 0, 1), (1, 1, 0) belong to this plane, this plane passes through the cube creating a pyramid the volume of which equals $1/6$. Therefore, the volume of the given region equals $8(1-1/6)=20/3$.

Question 4

Let A and B be two square matrices of the third order, all the elements of which are integers, and let AB = A + B. Find all the possible values of the determinant $|A-E|$ (E is an identity matrix of the third order).

Answer. ± 1.

Solution. Let us transform the given equation: $AB-A-B+E=E$, $(A-E)(B-E)=E$, therefore $|A-E|\cdot|B-E|=1$. Since the elements of these matrices are integers, $|A-E|$ and $|B-E|$ are also integers. Hence $|A-E|=|B-E|=1$ or $|A-E|=|B-E|=-1$. The first equation is satisfied, for example, when A = B = 2E, and the second one when A = B = 0.

Question 5

Let S be the set of all numbers of the form $2^x+2^y+2^z$ (x, y, z are pairwise distinct non-negative integers), recorded in ascending order. Find the 100-th element of this set.

Answer. 577.

Solution. Let us denote the desired number by N. It is obvious that S is a set of positive integers having exactly 3 unities in binary representation. The quantity of numbers which have exactly k unities and no more than n binary digits equals C_n^k.

Note that $C_9^3 = 84$, and $C_{10}^3 = 120$, and therefore $2^9 < N < 2^{10}$, or $N = 2^9 + 2^y + 2^z$, and N is the 16-th number of this kind. Since $C_6^2 = 15$, y = 6 and z = 0. Consequently, $N = 2^9 + 2^6 + 2^0$ =577.

Question 6

It is known that the transposition of a determinant can be seen as a symmetrical mapping relative to the main diagonal. How will the value of a determinant of n-th order change if it is symmetrically mapped relative to a secondary diagonal?

Ответ. The value of the determinant will not change.

Solution. We can obtain the mapping through a consecutive permutation of symmetrical rows and columns and the transposition of the determinant (these actions can be performed in any order). The total number of permutations of rows and columns is even since their number is equal, and therefore the value of the determinant will not change. (The geometric motif of the solution: sequential symmetrical mapping with respect to any two mutually perpendicular axes in a plane is a central symmetry).

Remark. It is also possible to find inductive proof by checking directly for n = 2 or n = 3, and performing an inductive step using the expansion of the determinant by rows (columns).

Question 7

On a line there are *n* disjoint segments, the length of each of which is equal to 3. Find all values of *n*, for which there exists *a* (regardless of the location of the segments), such that each of the segments contains a root of the equation: sin (x - a) = 0.

Answer. $1 \le n \le 22$.

Solution. Let us shift all of the segments along the straight line to a distance integrally divisible by π, so that their left ends will fall within the range $[0; \pi)$. If the obtained segments are not fully contained within this range, their parts lying within the range $[\pi; 2\pi)$ are to be shifted to a distance of π to the left. As a result, an "image" of each of the original segments will be a segment or a union of segments contained completely within the range $[0; \pi)$, and all the roots of the equation sin (x - a) = 0 will fall on the point $a \in [0; \pi)$. Let us call the part of the range $[0; \pi)$ that is not covered by a certain segment a "hole". The conditions of the problem for some *n* are satisfied if and only if there is a point common to all the "images", or, equivalently, if the union of all the "holes" does not coincide with the interval $[0; \pi)$. This implies that all the values of *n*, for which this is possible, satisfy the inequality $n(\pi - 3) < \pi$, i.e., $1 \le n \le 22$.

Question 8

Let $P(x)$ be a polynomial of degree *n*, such that $P(x) \ge 0$ for all real *x*. Prove that $f(x) = P(x) + P'(x) + P''(x) + ... + P^{(n)}(x) \ge 0$ for all *x*.

Solution. Note that $f(x) - f'(x) = P(x) \ge 0$ (since $P^{(n+1)}(x) = 0$ for all *x*). It follows from the conditions of the problem that $P(x)$ (and therefore also $f(x)$) are even degree polynomials with positive leading coefficients. Let us denote by x_0 the global minimum point of $f(x)$ for $x \in (-\infty;+\infty)$. Then $f(x) \geq f(x_0) \geq f'(x_0) \geq 0$ for all *x*.

Question 9

The function $f(x) = \dfrac{5x}{x^2+2x+2}$ is expanded into the Taylor series by powers of *x*: $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$. Find the sum of the coefficients of this series with numbers that are multiples of three: $\sum\limits_{k=0}^{\infty} a_{3k} = a_0 + a_3 + a_6 + \ldots$

Answer. 2.

Solution. Let us introduce the notation: $A = a_0 + a_3 + a_6 + \ldots$, $B = a_1 + a_4 + a_7 + \ldots$, $C = a_2 + a_5 + a_8 + \ldots$. Then $f(1) = 1 = = \mathrm{A} + \mathrm{B} + \mathrm{C}$.

Let us substitute x for one of the complex roots of the third degree of 1: $f\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = A + B\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) + C\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = A - \frac{B}{2} - \frac{C}{2} + (B-C)\frac{\sqrt{3}}{2}i$. On the other hand, $f\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 5\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = \frac{5}{2} + \frac{5\sqrt{3}}{2}i$, and therefore, $A - \frac{B}{2} - \frac{C}{2} = \frac{5}{2}$. Hence $\mathrm{A} = 2$.

(The radius of convergence of this series is $\sqrt{2}$ - this is the distance from 0 to the nearest singular point of function *f*(*x*) in the complex plane).

Question 10

At the beginning of a game, the number of cards shown in the figure below is placed on each square of a 10 × 10 board. On each turn, the players must select three adjacent squares located in the same row or column, and remove one card from each of the three squares. What is the minimum number of cards that will be left on the board at the end of the game?

10	9	8	7	6	5	4	3	2	1
20	19	18	17	16	15	14	13	12	11
30	29	28	27	26	25	24	23	22	21
40	39	38	37	36	35	34	33	32	31
50	49	48	47	46	45	44	43	42	41
60	59	58	57	56	55	54	53	52	51
70	69	68	67	66	65	64	63	62	61
80	79	78	77	76	75	74	73	72	71

90	89	88	87	86	85	84	83	82	81
100	99	98	97	96	95	94	93	92	91

Solution. First, let us subtract 1 from the squares 3-13-23, 33-43-53, 63-73-83 and 2 from the squares 4-14-24, 34-44-54, 64-74-84. Now all the squares in the rectangle from 2 to 84 contain the same number of cards, and since the width of the rectangle is 3 squares and the height - 9 squares, this means that we have 9 sets of 3 squares belonging to the same row/column, and therefore they can all be nullified.

10	9	8	7	6	5	4	3	2	1
20	19	18	17	16	15	14	13	12	11
30	29	28	27	26	25	24	23	22	21
40	39	38	37	36	35	34	33	32	31
50	49	48	47	46	45	44	43	42	41
60	59	58	57	56	55	54	53	52	51
70	69	68	67	66	65	64	63	62	61
80	79	78	77	76	75	74	73	72	71
90	89	88	87	86	85	84	83	82	81
100	99	98	97	96	95	94	93	92	91

Similarly, we can nullify another two rectangles the width of which is 3 squares and the height – 9. In other words, we subtract 2 from all the black squares in the second figure, and 1 from all the grey ones, and thus obtain sets of three squares containing the same number of cards each, which can then be nullified.

0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	11
0	0	0	0	0	0	0	0	0	21
0	0	0	0	0	0	0	0	0	31
0	0	0	0	0	0	0	0	0	41
0	0	0	0	0	0	0	0	0	51

0	0	0	0	0	0	0	0	0	61
0	0	0	0	0	0	0	0	0	71
0	0	0	0	0	0	0	0	0	81
100	99	98	97	96	95	94	93	92	91

Consequently, we are left with the third figure. Here we have 9×9 zeroes and some adjacent squares with non-zero numbers. Let us divide these squares into sets of three, as can be seen in the third figure, and subtract three cards from each set of three squares as many times as possible. Thus the numbers

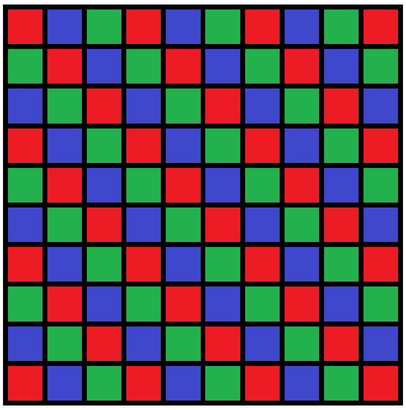
1, 11, 21, 31, 41, 51, 61, 71, 81, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100 are converted into numbers

1, 0, 10, 20, 0, 10, 20, 0, 10, 20, 0, 1, 2, 0, 1, 2, 0, 1, 2.

The sum of these remaining numbers is

$$1+30+30+30+3+3+3 = 1+90+9 = 100.$$

Can the game be played in such a way that fewer cards will be left? Let us color the board into 3 different colors, as shown in the figure below. All the squares are divided into sets of three, each set having one blue, one green and one red square. Hence, the difference between the number of cards in the red and the green squares, or the blue and the green squares, does not change. Note that at the end of our game all the green squares were nullified. Thus, the number of cards left on the green squares could not possibly be any smaller. And since the difference does not change, this means that the number of cards left on the blue and the red squares cannot possibly be any smaller either.



Question 11

a) Does there exist a sequence of digits infinite to the left ...... 625 such that every last n digits form an n-digit number x_n (possibly starting with zero) such that $x_n{}^2$ ends with x_n?

b) Prove that the sequence is aperiodic.

Solution.

a) Let us obtain x_n via the method of successive approximations. Suppose that we have already obtained x_n, and we now have to obtain x_{n+1}. Since $x_n{}^2$ ends with x_n, the number $x_n \cdot (x_n - 1)$ must be divisible by 10^n. Since the last digit of x_n is 5, x_n is divisible by 5 but not by 2, and $x_n - 1$ is divisible by 2 but not by 5. Consequently, x_n is divisible by 5^n, and $x_n - 1$ is divisible by 2^n. Suppose that *a* is the remainder of $x_n / 5^n$ modulo 5, *d* is the remainder of 2^n modulo 5, b is the remainder of $(x_n - 1)/\ 2^n$ modulo 2, e is the remainder of 5^n modulo 2. Let c be a digit with the remainder of a/d modulo 5, and b/e modulo 2. Then $x_{n+1} = x_n + c \cdot 10^n$ is the desired number, since x_{n+1} is divisible by 5^{n+1}, and at the same time $x_{n+1} - 1$ is divisible by 2^{n+1}.

(It is clear that e=1, but we conducted the reasoning in a generalized manner).

b) Let us suppose that the sequence is periodic. Let A be the period of length k, and B – the preperiod of length n. Then, the number $x_m = \underbrace{AA\ldots A}_{m \text{ раз}}B$ has the property that $x_m{}^2$ ends in x_m.

It can be easily seen that $x_m = B + 10^n A(10\ldots010\ldots01\ldots) = B + 10^n A \cdot \dfrac{10^{km} - 1}{10^k - 1}$ is congruent to $Z = B + \dfrac{10^n A}{10^k - 1}$ modulo 10^{n+km}. Hence $T = Z^2 - Z$ is congruent to zero modulo 10^{n+km} for any k. This would mean that T=0, which obviously is not true, since Z does not equal 0 or 1.

Remark. An idempotent is an element e, such that $e^2 = e$. In essence, it can be concluded from this problem that in a ring of 10-adic numbers Q_{10} the nontrivial idempotent is irrational, and that $Q_{10} = Q_2 + Q_5$.

Question 12

For non-negative a, b and c prove that:

$$\sqrt{a^2+b^2+c^2}+2\sqrt{ab+ac+bc} \geq \sqrt{a^2+2bc}+\sqrt{b^2+2ac}+\sqrt{c^2+2ab}$$

Proof: Let us perform the following transformations:

$$\sqrt{a^2+b^2+c^2}+2\sqrt{ab+ac+bc}-\left(\sqrt{a^2+2bc}+\sqrt{b^2+2ac}+\sqrt{c^2+2ab}\right)=$$

$$=\sum_{cyc}\left(\sqrt{a^2+b^2+c^2}-\sqrt{c^2+2ab}\right)-2\left(\sqrt{a^2+b^2+c^2}-\sqrt{ab+ac+bc}\right)=$$

$$=\left|\sum_{cyc}\sqrt{c^2+2ab}=\sqrt{c^2+2ab}+\sqrt{a^2+2bc}+\sqrt{b^2+2ca}\right|=$$

$$=\sum_{cyc}\frac{(a-b)^2}{\sqrt{a^2+b^2+c^2}+\sqrt{c^2+2ab}}-\sum_{cyc}\frac{(a-b)^2}{\sqrt{a^2+b^2+c^2}+\sqrt{ab+ac+bc}}=$$

$$=\sum_{cyc}\frac{-(a-b)^2(c-a)(c-b)}{\left(\sqrt{a^2+b^2+c^2}+\sqrt{c^2+2ab}\right)\cdot\left(\sqrt{a^2+b^2+c^2}+\sqrt{ab+ac+bc}\right)\cdot\left(\sqrt{ab+ac+bc}+\sqrt{c^2+2ab}\right)}=$$

$$=\sum_{cyc}\frac{-(a-b)^2(c-a)(c-b)}{\left(\sqrt{a^2+b^2+c^2}+\sqrt{c^2+2ab}\right)\cdot\left(\sqrt{a^2+b^2+c^2}+\sqrt{ab+ac+bc}\right)\cdot\left(\sqrt{ab+ac+bc}+\sqrt{c^2+2ab}\right)}+$$

$$+\sum_{cyc}\frac{(a-b)^2(c-a)(c-b)}{2\left(\sqrt{a^2+b^2+c^2}+\sqrt{ab+ac+bc}\right)^2\cdot\sqrt{ab+ac+bc}}=$$

$$=\sum_{cyc}\frac{(a-b)^2(c-a)(c-b)\left(\sqrt{a^2+b^2+c^2}+\sqrt{c^2+2ab}\right)}{\left(\sqrt{a^2+b^2+c^2}+\sqrt{c^2+2ab}\right)\left(\sqrt{a^2+b^2+c^2}+\sqrt{ab+ac+bc}\right)^2}\times$$

$$\times\frac{\left(\sqrt{ab+ac+bc}+\sqrt{c^2+2ab}\right)}{\left(\sqrt{ab+ac+bc}+\sqrt{c^2+2ab}\right)\sqrt{ab+ac+bc}}-\sum_{cyc}\frac{2(a-b)^2(c-a)(c-b)}{\left(\sqrt{a^2+b^2+c^2}+\sqrt{c^2+2ab}\right)}\times$$

$$\times\frac{\left(\sqrt{a^2+b^2+c^2}+\sqrt{ab+ac+bc}\right)\sqrt{ab+ac+bc}}{\left(\sqrt{a^2+b^2+c^2}+\sqrt{ab+ac+bc}\right)^2\left(\sqrt{ab+ac+bc}+\sqrt{c^2+2ab}\right)\sqrt{ab+ac+bc}}=$$

Here we utilize the fact that

$$\sum_{cyc}\frac{(a-b)^2(c-a)(c-b)}{2\left(\sqrt{a^2+b^2+c^2}+\sqrt{ab+ac+bc}\right)^2\cdot\sqrt{ab+ac+bc}}=0$$

and that

$$\left(\sqrt{a^2+b^2+c^2}+\sqrt{c^2+2ab}\right)\left(\sqrt{ab+ac+bc}+\sqrt{c^2+2ab}\right)-$$
$$-2\left(\sqrt{a^2+b^2+c^2}+\sqrt{ab+ac+bc}\right)\sqrt{ab+ac+bc}=$$
$$=c^2+2ab-(ab+ac+bc)+\left(\sqrt{a^2+b^2+c^2}+\sqrt{ab+ac+bc}\right)\left(\sqrt{c^2+2ab}-\sqrt{ab+ac+bc}\right)=$$
$$=(c-a)(c-b)\left(1+\frac{\sqrt{a^2+b^2+c^2}+\sqrt{ab+ac+bc}}{\sqrt{ab+ac+bc}+\sqrt{c^2+2ab}}\right).$$