1. The road from point A to point B goes uphill for $3-\sqrt{2}$ km, downhill for $2+\sqrt{3}$ km, and is horizontal for the rest of the way. A traveler covered the distance from A to B and back again in 4 hours. He walked uphill at a speed of 3 km/h, downhill at a speed of 6 km/h, and horizontally at a speed of 4 km/h. What is the length of the road from A to B (in kilometers)?

Answer: 8.

Solution. Let us denote the length of the uphill, downhill and horizontal sections of the road as x, y and z respectively. According to the question:

$$\frac{x+y}{3}+\frac{x+y}{6}+\frac{2z}{4}=4.$$

By transforming the left-hand side of the above equation we obtain:

$$\frac{1}{2}(x+y+z)=4,$$

Hence $x+y+z=8$.

2. The regular 2011-gon of side 1 circumscribes a circle and is inscribed in a circle. Find the area of the ring between the two circles.

Answer: $\pi/4$.

Solution. We denote as R and r the radii of the inscribed and circumscribed circles, respectively. The area of the ring then is $S=\pi(R^2-r^2)$. On the other hand, it follows from the Pythagorean Theorem that $R^2-r^2=(1/2)^2$. Therefore $S=\pi/4$.

3. In a convex n-gon no three diagonals intersect at one point. What is the number of points of intersection of the diagonals?

Answer: $\dfrac{n(n-1)(n-2)(n-3)}{24}$.

Solution. For any four vertices of a polygon there exists exactly one pair of intersecting diagonals that join any two these vertices. Therefore, the required number of intersection points is equal to $\binom{n}{4}$.

4. Find the 300-th digit after the decimal point of the number $\sqrt[3]{0.999\ldots9}$ (a hundred nines).

Answer: 5.

Solution. Let us write the Taylor formula at zero for the function $f(x)=(1-x)^{\frac{1}{3}}$ with the remainder in Lagrange form:

$$(1-x)^{\frac{1}{3}}=1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}(1-\xi)^{-\frac{8}{3}}x^3 \quad (0<\xi<x).$$

By substituting the value $x = 10^{-100}$, into the above formula, we find that the sum (without the remainder) is equal to $0.99\ldots966\ldots655\ldots$ (after the decimal point there are 100 nines, then 100 sixes, and then — fives). The remainder is easy to estimate. Its absolute value is less than 10^{-301}, and therefore it has no effect on the first 300 digits after the decimal point.

5. Calculate the sum of the series:

$$\sum_{n=1}^{\infty} \frac{n}{(n+2)!}.$$

Answer: $3 - e$.

Solution. Let us utilize the fact that

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e \quad \text{and} \quad \frac{n}{(n+2)!} = \frac{1}{(n+1)!} - 2\frac{1}{(n+2)!}.$$

Thus we obtain

$$\sum_{n=1}^{\infty} \frac{n}{(n+2)!} = \sum_{n=1}^{\infty} \frac{1}{(n+1)!} - 2\sum_{n=1}^{\infty} \frac{1}{(n+2)!} = \sum_{n=2}^{\infty} \frac{1}{n!} - 2\sum_{n=3}^{\infty} \frac{1}{n!} = (e-2) - 2(e-2.5) = 3 - e.$$

6. When I cross the street in a place other than the pedestrian crossing, the probability of being hit by a car is 0.01. What is the probability of being hit by a car if I cross the street in such a place 100 times?

Answer: $1 - 1/e$

Solution. The probability of crossing the street safely once is 0.99, and of crossing the street safely 100 times is 0.99^{100}. Thus, the desired probability is approximately equal to

$$\lim_{x \to 0} (1-x)^{\frac{1}{x}} = 1/e.$$

Let us now prove that the margin of error of this estimate is no greater than 0.01. Note that $(1-x)^{\frac{1}{x}}$ is a monotonically decreasing function, while $(1-x)^{\frac{1}{x}+1}$ is monotonically increasing on the interval $(0, 1)$, and they both have a common limit equal $1/e$. This means that,

$$0.99^{101} < 1/e < 0.99^{100},$$

and hence

$$0.99^{100} - 1/e < 0.99^{100} - 0.99^{101} = 0.99^{100} \cdot 0.01 < 0.01.$$

7. Find a matrix A, such that

$$A^{2011} = \begin{pmatrix} -4021 & -4022 \\ 4022 & 4023 \end{pmatrix}$$

Answer: $\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}$.

Solution. Let us transform the matrix on the right-hand side to the Jordan normal form. Thus we obtain a Jordan block $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and the transition matrix $\begin{pmatrix} -1 & \frac{1}{4022} \\ 1 & 0 \end{pmatrix}$ The root of the 2011-th power of the Jordan cell is extracted explicitly: $\begin{pmatrix} 1 & \frac{1}{2011} \\ 0 & 1 \end{pmatrix}$. Returning to the original basis, we obtain the answer: $\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}$.

8. Evaluate the integral with error of no more than 20%.

$$\int_{-\pi/2}^{\pi/2} \sin^{100} x dx.$$

Answer: 0.25.

Solution. Let us use the following approximate formulas, valid for small values of x:

$$\begin{aligned} \cos x &\sim 1 - \frac{x^2}{2}; \\ \ln(1+x) &\sim x; \\ e^x &\sim 1 + x, \end{aligned}$$

as well as the formula for the Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Thus we obtain

$$\int_0^\pi \sin^{100} x dx = \int_{-\pi/2}^{\pi/2} \cos^{100} x dx = \int_{-\pi/2}^{\pi/2} e^{100 \ln \cos x} dx \sim \int_{-\pi/2}^{\pi/2} e^{-50x^2} dx =$$
$$= \frac{1}{5\sqrt{2}} \int_{-5\sqrt{2}\pi/2}^{5\sqrt{2}\pi} e^{-t^2} dt \sim \frac{1}{5\sqrt{2}} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{\sqrt{2\pi}}{10} = 0.250\ldots$$

Let us show that the function

$$f(x) = \frac{\cos x}{e^{-x^2/2}}$$

decreases at $x > 0$. It's derivative has the form

$$f'(x) = \frac{x\cos x - \sin x}{e^{-x^2/2}}$$

The condition $f'(x) < 0$ is equivalent to $x < \tan x$, which is satisfied at $x > 0$. Since $f(0) = 1$, we have $\cos x < e^{x^2/2}$ at $0 < x < \pi/2$.

Now let us estimate the margin of error caused by 2 approximations. Let's start with the error caused by changing the domain of integration. We have

$$2\int_{5\sqrt{2}\pi/2}^{\infty} e^{-t^2} dt < 2\int_{10}^{\infty} e^{-t} dt < 2e^{-10} < 2^{-9} < 0.002,$$

which is obviously less than 1% of the answer.

Let's estimate margin of error caused by substituting e^{-50x^2} instead of $\cos^{100} x$. Let us estimate the margin of error inside and outside the neighbourhood of zero separately. As the neighbourhood let us consider the interval $(-0.3, 0.3)$.

Outside the interval we have

$$2\int_{0.3}^{\pi/2} e^{-50x^2} dx < (\pi - 0.6)e^{-4.5} = 0.0282\ldots$$

which is less than 11,5%.

Inside the interval the substitution of $\cos^{100} x$ for e^{-50x^2} causes relative margin of error no more than

$$1 - \frac{\cos^{100} 0.3}{e^{50\cdot 0.3^2}} < 0.075$$

So the relative margin of error of the integral is also no more than 7.5%.

Thus the overall margin of error is less than $1\% + 11.5\% + 7.5\% = 20\%$.