1. Let $f(x) = \frac{1}{x^2+1}$. Find $f^{(6)}(0)$ (the value of the sixth derivative of the function $f(x)$ at zero).

   Answer: -720.

   Solution: We expand the given function into a Taylor series at the point $x = 0$:
$$f(x) = 1 - x^2 + x^4 - x^6 + \cdots$$
   The coefficient of x^6 equals -1, but it also equals $f^{(6)}(0)/6!$. And thus we obtain the answer:$f^{(6)}(0) = -6!$.

2. A mushroom is called *bad* if there are 10 or more worms in it. A basket contains 91 *bad* mushrooms and 10 *good* mushrooms. Is it possible that after some of the worms crawl off the *bad* mushrooms and onto the *good* ones, all of the mushrooms will become *good*?

   Answer: It is not possible.

   Solution: Since a *bad* mushroom has at least ten worms and the basket contains 91 *bad* mushrooms, there must be at least 910 worms in the basket. The total number of mushrooms in the basket is 101. If all of the mushrooms in the basket became good, it would mean that none of them would have more than 9 worms, and therefore, the total number of worms could be no greater than 909. But since the total number of worms must remain the same as it was at the beginning, this situation is impossible.

3. Suppose that M is a point on side AB of triangle ABC, such that $AM : MB = 2$, and N is a point on side BC of the same triangle, such that $BN : NC = 2$. Let X denote the point of intersection of segments CM and AN. Find $AX : XN$ and $CX : XM$.

   Answer: $AX : XN = 6$, $CX : XM = \frac{3}{4}$.

   Solution. There are two possible solutions:

   1) Let us introduce the notations $\vec{u} = \overrightarrow{BA}$, $\vec{v} = \overrightarrow{BC}$ and expand the vector $\overrightarrow{BX}$ in $(\vec{u}, \vec{v})$ basis in two ways. On the one hand, $\overrightarrow{AN} = -\vec{u} + \frac{2}{3}\vec{v}$ and therefore, $\overrightarrow{AX} = \alpha\overrightarrow{AN} = -\alpha\vec{u} + \frac{2}{3}\alpha\vec{v}$ for some α, and thus, $\overrightarrow{BX} = \vec{u} + \overrightarrow{AX} = (1-\alpha)\vec{u} + \frac{2}{3}\alpha\vec{v}$. On the other hand,$\overrightarrow{CM} = \frac{1}{3}\vec{u} - \vec{v}$

   and therefore,$\overrightarrow{CX} = \beta\overrightarrow{CM} = -\frac{1}{3}\beta\vec{u} - \beta\vec{v}$ for β, and thus, $\overrightarrow{BX} = \vec{v} + \overrightarrow{CX} = \frac{1}{3}\beta\vec{u} + (1-\beta)\vec{v}$.

By equating the coefficients of these two linear representations we obtain a linear system of equations for α and β:

$$\begin{cases} 1-\alpha = \frac{1}{3}\beta \\ \frac{2}{3}\alpha = 1-\beta \end{cases},$$

by solving which we find that , $\alpha = \frac{6}{7}$, $\beta = \frac{3}{7}$. And from this we can easily obtain the desired ratios: $AX : XN = \frac{\alpha}{1-\alpha} = 6$, $CX : XM = \frac{\beta}{1-\beta} = \frac{3}{4}$.

2) Let us place different weights at the three apexes of the triangle: a weight of 1 at point A, a weight of 2 at point B, and a weight of 4 at point C. We will now find the center of mass of the resulting system of material points in two different ways. Firstly, we can substitute the weights at points A and B for a single weight the mass of which is 3, situated at their center of mass point M. Therefore, the center of mass must be situated on the segment CM and it must divide it into two segments the ratio of which is $3 : 4$. Similarly, by substituting the weights at points B and C for a single weight of 6 situated at their center of mass point N, we find that the center of mass must be situated on the segment AN and that it divides this segment into two segments the ratio of which is 6:1. Therefore the center of mass has to be situated at the intersection of the segments CM and AN which is point X, and it must divide these segments according to the ratios given in the answer.

4. It is known that function $f(x)$ is even, and that function $g(x) = f(2011 - x)$ is odd. Prove that $f(x)$ is a periodic function and find its period.

   Answer: 8044.

   Solution.

   According to the problem $f(x + 4022) = f(2011 - (-x - 2011)) = g(-x - 2011) = -g(x + 2011) = -f(2011 - (x + 2011)) = -f(-x) = -f(x)f(x+4022) = -f(x)$. Thus, $f(x+8044) = -f(x+4022) = f(x)$. It follows that $f(x+8044) = -f(x+4022) = f(x)$. And therefore 8044 is the period of function $f(x)$.

   Note: Any multiple of 8044 will also be a period of function $f(x)$.

5. One of the faces of a unit cube has a circle inscribed in it, and another face of the same unit cube is circumscribed by a circle. Find the shortest distance between two points that lie on the circumferences of these two circles.

   Answer: $(\sqrt{3} - \sqrt{2})/2$.

   Solution.

   Let us construct two concentric spheres, the centers of both of which coincide with the center of the cube. The surface of the first of these spheres contains the circumference of the inscribed (smaller) circle, and the surface of the second contains the circumference of the circumscribed (larger) circle. The projection of the circumference of the smaller circle on the surface of the larger sphere taken from the center of the cube intersects the circumference of the larger circle. It follows that there exists a ray the initial point of which is the center of the cube, which intersects the circumferences of both circles. The distance between the points of intersection obviously equals the difference of the radii of the spheres, and this difference, which equals $(\sqrt{3} - \sqrt{2})/2$, is the answer to the question.

6. Let A and B be square matrices of the third order, and furthermore, let all the elements of matrix B equal unity. It is known that $detA = 1$, $det(A + B) = 1$. Find $\det(A + 2011 \cdot B)$.

   Answer: 2011.

   Solution:

   Let us examine the function $f(x) = \det(A + x \cdot B)$. We can transform the determinant by subtracting the first row from the second and third rows, and expand the resulting determinant along the first row. It is obvious that $f(x)$ is a linear function, i.e. $f(x) = ax + b$. Since $a = 1$, $b = 0$, it follows that $a = 1$, $b = 0$. Therefore, $f(x) = x$ and $\det(A + 2011 \cdot B) = f(2011) = 2011$.

7. A particle is moving along a straight line. The direction of its movement can change, but its acceleration at any given moment does not exceed $1\ m/sec$ in absolute value. One second after it begins moving the particle returns to its starting point. Prove that its speed 0.5 sec after it begins moving is no greater than 0.25 m/sec.

Solution:

In the following calculations it is implied that all the time intervals are expressed in seconds and all the intervals of length measured in meters. Let $v(t)$ be the speed and $a(t)$ the acceleration of the given particle at time t. According to the problem $|a(t)| = |v'(t)| \leqslant 1$ and $\int_0^1 v(t)dt = 0$. We must estimate $v(0.5)$. In order to do this, let us examine its absolute value:

$$|v(0.5)| = |v(0.5) - 0| = \left| v(0.5) - \int_0^1 v(t)dt \right| =$$
$$= \left| v(0.5) \int_0^1 dt - \int_0^1 v(t)dt \right| = \left| \int_0^1 (v(0.5) - v(t))\, dt \right|.$$

Let us use Lagranges theorem: $v(0.5) - v(t) = v'(c)(0.5 - t)$, where $c(t) \in [0.5; t]$. Thus

$$|v(0.5)| = \left| \int_0^1 v'(c)(0.5 - t)dt \right| \leqslant \int_0^1 |v'(c)(0.5 - t)|\, dt =$$
$$= \int_0^1 |v'(c)| \cdot |(0.5 - t)| dt.$$

But $|v'(t)| \leqslant 1$. Therefore,

$$|v(0.5)| \leqslant \int_0^1 |0.5 - t| dt = \int_0^{0.5} (0.5 - t)dt + \int_{0.5}^1 (t - 0.5)dt = \frac{1}{4}.$$

8. Does a real function $f(x)$, which is defined on the entire real axis, and such that $f(f(x)) = -x^{2011}$ for every $x \in \mathbb{R}$ exist?

   Answer: It does not.

   Solution:

   Let us assume the opposite, i.e. that such a function does exist. Let $s = f(0)$. Then $f(s) = f(f(0)) = -0^{2011} = 0$. In addition $s = f(0) = f(f(s)) = -s^{2011}$. Therefore s = 0 and thus s(0) = 0. Let $u = f(1)$. Then $f(u) = f(f(1)) = -1^{2011} = -1$. Let $v = f(1)$. This implies that $f(f(u)) = v$, $f(f(v)) = u$. Therefore, $f(f(f(f(u)))) = u$, i.e.$u^{2011^2} = u$. And it then follows that $u = 0$, 1 or -1. Let us examine each of these three cases separately. The equality $u = 0$ contradicts the following two equalities: $f(u) = 1$ and $f(0) = 0$. If $u = 1$, then since $u = f(1)$, we obtain $f(1) = 1$ and $f(f(1)) = 1$. This too is a contradiction. If $u = 1$, then since $f(u) = 1$, we obtain $f(1) = 1$ and $f(f(1)) = 1$. Once again, a contradiction.

9. Solve the following equation: $|2^x - 3^y| = 1$. It is known that x and y are positive integers.

   Answer: $x = 1, y = 1; x = 2, y = 1; x = 3, y = 2.$

   Solution:

   For $x = 1$ and or $x = 2$, the solutions can be found easily. Next we consider the case where $x \geqslant 3$. Let us examine separately the case where the argument of the absolute value is positive and the case where it is negative. First case: $2^x - 3^y = 1$. In this case $3^y \equiv -1 \mod 8$, which is impossible. Second case: $2^x - 3^y = -1$. In this case $3^y \equiv -1 \mod 8$, which means that y is an even number. Therefore, $2^x = (3^{y/2} + 1)(3^{y/2} - 1)$. Each of the two factors on the right side of this equality must be a power of two, which is possible only for $y = 2$.

10. In the Republic of Anchuria a political party that has more than one member can exist for no longer than one day. The following day the party splits into two fractions, and each of those declare themselves a new party. Whenever a new party is created each of its members receives a membership card. On a certain day 2011 citizens of this republic created a new political party. After a while, following numerous splits, 2011 parties consisting of one member each were created. What are the minimal and the maximal number of membership cards that could have been issued during this entire period of time?

    Answer: 24095 and 2025076.

    Solution:

    Let us denote the number of membership cards that were issued by $h(n)$ for the case where n was the initial number of party members and each split resulted in two maximally unequal factions (i.e., when one of the resulting factions consisted of only one member). Similarly, we denote the number of membership cards by $l(n)$ for the case where each split resulted in two maximally equal factions (i.e., the party either split into two equal factions, or the difference between the number of members in each of the two factions was one, depending on whether the number of members in the party was even or odd). This definition implies that the functions $h(n)$ and $l(n)$ are defined by the following recurrence relations:

$$h(1) = 1;\; h(n) = h(1) + h(n-1) + n \text{ , when } n \geqslant 2;$$
$$l(1) = 1;\; l(n) = l(\lfloor n/2 \rfloor) + l(\lceil n/2 \rceil) + n \text{ , when } n \geqslant 2,$$

where $\lfloor x \rfloor$ is the greatest integer that does not exceed x, and $\lceil x \rceil$ is the smallest integer that is greater or equal to x. We can now find explicit formulas for functions $h(n)$ and $l(n)$:

$$h(n) = \frac{n^2 + 3n - 2}{2}, \quad l(n) = (r+1)n + 2s,$$

(in the above equations and hereafter r and s are uniquely determined non-negative integers, such that $n = 2^r + s$ and $0 \leqslant s < 2^r$). They can be easily proved by mathematical induction.

Let us also introduce the following two functions:

$$h'(n) = h(n+1) - h(n) = n + 2,$$
$$l'(n) = l(n+1) - l(n) = r + 3.$$

Note that both of these functions $l'(n)$ and $h'(n)$ are non-decreasing. Let us now prove by induction that $l(n)$ and $h(n)$ are, respectively, the minimal and the maximal number of membership cards issued if the initial party consisted of n members. The basis of the induction is obvious. The inductive step can be reduced to proving that the inequalities

$$h(n_1) + h(n_2) \leqslant h(1) + h(n-1),$$
$$l(n_1) + l(n_2) \geqslant l(\lfloor n/2 \rfloor) + l(\lceil n/2 \rceil)$$

hold, provided that $n_1 + n_2 = n$. Let us prove a more general statement, namely that if $n_1 + n_2 = \text{const}$, then the expressions $h(n_1) + h(n_2)$ and $l(n_1) + l(n_2)$ and the difference $|n_2 - n_1|$ increase simultaneously. In order to show that this is true it is sufficient to prove that the following inequalities

$$h(n_1) + h(n_2) \leqslant h(1) + h(n-1),$$
$$l(n_1) + l(n_2) \geqslant l(\lfloor n/2 \rfloor) + l(\lceil n/2 \rceil)$$

hold, provided that $n_1 \leqslant n_2$.

These inequalities can be reduced to the following identical inequalities:

$$h'(n_1 - 1) \leqslant h'(n_2),$$
$$l'(n_1 - 1) \leqslant l'(n_2),$$

which hold by virtue of the already proven monotonicity of the functions $h'(n)$ and $l'(n)$. Therefore, the minimal number of membership cards that could have been issued equals $l(2011) = 24095$, and the maximal number equals $h(2011) = 2025076$.