Team Olympiad

1. A batch of 100 tons of melons was sent from Haifa to Eilat. On the way there shrinkage occurred. The dry matter content in Haifa was 2%, and in Eilat 2.5%. How many tons of melons arrived in Eilat?

Answer: 80 tons.

Solution. The total amount of dry matter is 2 tons. That is 2.5% of 80 tons.

2.

$$f(x) = \ln(x + \sqrt{1 + x^2}).$$

Find $f^{(7)}(0)$. ($f^{(n)}(0)$ is the value of the n-th derivative of function f at zero.)

Solution.

$\left(\ln(x + \sqrt{1 + x^2})\right)' = \frac{1}{\sqrt{1+x^2}}$. Using the binomial theorem we expand $\frac{1}{\sqrt{1+x^2}}$ into a Taylor series:

$\frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-1/2} = 1 - x^2/2 + 1/2 \cdot 3/2x^4/2 - 1/2 \cdot 3/2 \cdot 5/2x^6/6 + ...$

$((a + b)^n = \sum_{k=0}^{\infty} a^{n-k}b^k\binom{n}{k}$, we have $n = -1/2, a = 1, b = x^2; \binom{n}{k} = \frac{-1/2 \cdot -3/2 \cdots -(2k-1)/2}{k!}$.

3. The sum of the absolute values of elements of the integer matrix 3×3 A is 3. What is the minimal possible sum of the moduli of the matrix elements of A^{100}, if the matrix is not equal to zero? (A matrix is a table of numbers called *matrix elements*).

Answer: 1.

Example:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

4. In a unit cube a closed polygonal chain connects all of its faces. What is the minimum possible length of this chain?

Answer: $2\sqrt{3}$.

Solution. Let us place the origin at one of the vertices of the cube and place the axes on its edges. Assume that a polygonal chain with n segments is a trajectory of the flight of a fly. Let (x_i, y_i, z_i), $i = 1, 2, \ldots, n$ be the coordinates of the vectors that define the edges of the polygonal chain. Then $d_i = \sqrt{x_i^2 + y_i^2 + z_i^2}$ is the length of its i-th segment, and $D = d_1 + d_2 + \ldots + d_n$ is the length of the polygonal chain. Suppose that $X = |x_1| + \ldots + |x_n|$ (This is the way of projecting the trajectory of the fly onto axis Ox). Similarly, $Y = |y_1| + \ldots + |y_n|$, $Z = |z_1| + \ldots + |z_n|$. We can consider that in the beginning the fly was in the plane $x = 0$ (the plane of one of the faces), then it moved to the plane $x = 1$ of the opposite face and then came back. This means that $X \geq 2$. Similarly, $Y \geq 2$ and $Z \geq 2$.

The sum of the vectors with coordinates $(|x_i|, |y_i|, |z_i|)$ is equal to vector (X, Y, Z). Since the sum of the lengths of these vectors is no smaller than the length of their sum, we have: $d_1 + d_2 + \ldots + d_n \geq \sqrt{X^2 + Y^2 + Z^2} \geq 2\sqrt{3}$.[1]

An example of a trajectory of length $2\sqrt{3}$ is the main diagonal of the cube, traversed twice. A closed trajectory of the same length will be obtained if the fly begins to move from any point within the cube in the direction parallel to the main diagonal of the cube, and will be "reflected specularly" from each of the faces of the cube. (Prove this!)

5. $S(n)$ is the sum of the digits of number n. Let $S(n) = 3$, $S(n^3) = 27$. Find the minimal possible number n.

Answer: 111.

Solution. The following lemma can be easily proven.

Lemma . For any natural numbers m and n the following inequality $S(m, n) \leq S(m) \cdot S(n)$ holds, and the equality $S(m, n) = S(m) \cdot S(n)$ holds iff the "carry" operation is not performed when m is multiplied by n (in other words, an extra digit is not "carried" into the next column).

It follows from the Lemma that $S(n)^3 = S(n^3)$ holds only if the "carry" operation in not performed when n is raised to power 3. The condition is satisfied only if all the digits of n are less than 2 or one of the digits equals 2 and all the others equal zero.

6. A right triangle T of unit area is given. it A line is called a tangent if it has exactly one common point with T - *the point of tangency.* From point x_0 a tangent is drawn to T, and then the point x_0 is reflected symmetrically to the point of tangency, and the point x_1 is thus obtained. (If the point x_0 lies on the extension of a side, the point x_1 is not defined.) From point of x_1 a tangent is drawn to T not passing through x_0, and then the point x_1 is reflected symmetrically to the point of tangency, and thus the point x_1 is obtained, etc. What is the area of the set of points of period 6 (i.e., when $x_0 = x_6$, and all x_i are defined)?

Answer: 18.

Solution. Consider three regular hexagons U, V, S, adjacent to the sides of T. Their centers P, Q, R form a trajectory of period 3, and the subsequent reflections about the vertices of T form central symmetries with centers at the points P, Q, R. Triple reflection leads to a central symmetry, and a six-fold reflection - to the identity transformation. It remains to note that the evolution of points from U, V, S have the same structure and that other points have a greater period.

7. Find the 301-st digit after the decimal point of $1 + \ln(0.99\ldots9)$ (100 nines).

Answer: 6.

[1] The reader can think about how the calculations in the above solution can be replaced by geometric reasoning using the reflection of the trajectory relative to the planes of the faces.

Solution. $1+\ln(0.99\ldots9) = 1+\ln(1-10^{-100})$. We expand $1+\ln(1-x) = 1 - x - x^2/3 - x^3/3 - x^4/4 - \ldots$. We substitute $x = 10^{-100}$. And we obtain

$$1+\ln(1-10^{-100}) = 0.(9) - 10^{-100} - 10^{-200}/2 - 10^{-300} \cdot 0.(3) - 10^{-400}/4 - \ldots$$

$0.(9) = 0.9999\ldots, 0.(3) = 0.333\ldots$

Hence, we conclude that from a fraction which, starting with the 202-nd digit after the decimal point, has only nines in its period (i.e., $0.(9) - 10^{-100} - 10^{-200}/2$), we must subtract a fraction which, starting with the 301-st digit after the decimal point, has only threes in its period and before that - zeros (i.e., $10^{-300}/3$). The remaining terms have digits with numbers greater than 400 (the n-th term is less than 10^{-n}), and their sum is smaller than 10^{-400} in absolute value, and therefore all of them together will not affect the 301-st digit.

8. The sum of the absolute values of elements of the integer matrix 2×2 A is 3. What is the minimal possible sum of the moduli of the matrix elements of A^{100}, if the matrix is not equal to zero? (A matrix is a table of numbers called *matrix elements*).

Answer: 2.

Example:

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

Let us demonstrate that the sum of the elements of A^{100} cannot equal 1.

Otherwise, A^{100} would equal $\begin{pmatrix} 0 & \pm1 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ \pm1 & 0 \end{pmatrix}$ or $\begin{pmatrix} \pm1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & \pm1 \end{pmatrix}$. In the first two cases we have a nilpotent 2×2 matrix with square zero, which does not have a square root (and obviously does not have a 100th degree root). In the latter cases we have a diagonal matrix one of the eigenvalues of which is 0, and the other ±1. Its 100th degree root has one eigenvalue 0, and another - 100th degree root of ±1, and besides it is diagonal. If this root is integral, then it takes the form $\begin{pmatrix} \pm1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & \pm1 \end{pmatrix}$. In any case, the sum of the matrix elements equals 1 and not 3. It can be easily seen that the modules of the eigenvalues of A cannot exceed 1. If the matrix elements are three unities then this is possible only when the eigenvalues are roots of the third or sixth degree of 1. The presence of 2 or 3 is examined separately.