Solutions

1. **Answer:** This is not true.

   Let us evaluate the concentration of sugar in a mixture of V_1 liters of concentration c_1 and V_2 liters of concentration c_2. We will then have V_1+V_2 liters of mixture containing $c_1V_1+c_2V_2$ liters of sugar. Therefore the concentration of sugar in the mixture is

   $$c = c_1 \cdot \frac{V_1}{V_1 + V_2} + c_2 \cdot \frac{V_2}{V_1 + V_2}$$

   (this result has the following geometric interpretation: the point c on the coordinate line divides the segment $[c_1, c_2]$ by the ratio $V_2 : V_1$). Therefore, by varying volumes one can get any result between c_1 and c_2.

   Back to our problem, first set the four concentrations of sugar subject to the given inequalities and such that the juice in the third glass be sweeter than that in the second one. Now, if the capacities of the first and the fourth glasses are substantially smaller than those of the third and the second respectively, then the mixture of the first and the third juices will be about as sweet as the third one was in the beginning, while the mixture of the second and the fourth juices will be about as sweet as the second one was in the beginning. It is now easy to build the required counterexample:

   1st glass — 1 l, 60%,
   2nd glass — 3 l, 50%,
   3rd glass — 3 l, 20%,
   4th glass — 1 l, 10%.

   Mixing the 1st and the 3rd glasses, we get 4 liters of juice with 30% concentration of sugar. Mixing the 2nd and the 4th glasses, we get 4 liters of juice with 40% concentration of sugar.

2. **Answer:** $4ab$.

   Add two more lines to the drawing: $x = 2a$ and $y = 2b$. These 2 lines and the two coordinate axes divide the circle into 9 parts: the central one, the 4 corners (of equal areas, by symmetry), and two more pairs – (the top one, the bottom one) and (the right one, the left one), where in every pair the areas of its two elements are equal. These equalities yield that the expression to be evaluated is equal to the area of the central part.

3. **Answer:** 16 liters.

   $E(v)$ is the number of kilometers covered by a vehicle traveling with constant velocity v per one liter of fuel. Therefore its reciprocal $\frac{1}{E(v)}$ is the number of liters of fuel required for a ride of one kilometer with constant velocity v.

   Riding for the time T with constant velocity v one would cover the distance vT and burn
$$\frac{1}{E}S = \frac{vT}{E(v)}.$$
   liters of fuel. It is given that the velocity is changing with the time so the fuel consumption is given by the integral

$$\int_0^T \frac{v(t)\,dt}{E(v(t))} = \int_0^1 \frac{168v(t)\,dt}{\sqrt{v^3(t)}} = 168\int_0^1 \frac{dt}{\sqrt{v(t)}} =$$
$$168\int_0^1 \frac{dt}{\sqrt{144\sqrt[4]{t}}} = 14\frac{8}{7}\,t^{\frac{7}{8}}\Big|_0^1 = 16\ (l).$$

4. **Answer:** The sequence converges for any given values of the parameters, the limit is $\sqrt[m]{a}$.

   By the given recurrence, x_{n+1} is the arithmetic mean of m numbers, of which all but the last are x_n and the last one is $\dfrac{a}{x_n^{m-1}}$. The geometric mean of these numbers is $\sqrt[m]{a}$, therefore by the Cauchy inequality of arithmetic and geometric means all terms of the sequence starting from x_2 are greater or equal than $\sqrt[m]{a}$.

   Since all terms of the m-tuple but the last one are not less than its geometric mean, the last term does not exceed it. Therefore, the arithmetic mean x_{n+1} of the terms of the m-tuple does not exceed x_n (for $n \geq 2$). We showed that the sequence is non-increasing and bounded from below. Therefore it converges by the Bolzano–Weierstrass theorem.

Denote its limit by x and let n go to infty in the recurrence to get

$$x = \frac{(m-1)x + \dfrac{a}{x^{m-1}}}{m},$$

hence $x = \sqrt[m]{a}$.

Remark. We have just developed a method of evaluation of the mth root of a number using solely the four arithmetic operations.

5. Divide the floor area into 1×1 squares and mark those in the odd positions of the odd rows. Note that every 1×4 tile always covers an even number of marked squares (two or zero), while every 2×2 tile always covers exactly one marked square. Therefore if the whole floor area is covered, then the number of 2×2 tiles has the same parity (odd/even) as the total number of marked squares. Hence, if the total number of 2×2 tiles now differs from the original one by an odd quantity, then the floor cannot be covered with the available tiles.

6. **Answer:** $\frac{4}{3}\pi R^3 \simeq 33510321$.

   Denote the number of points by N. Let each of the lattice points inside the square be the center of a unit cube. The union of all these cubes contains the ball of radius $R - \sqrt{3}/2$ and is contained in the ball of radius $R + \sqrt{3}/2$. The volumes of the balls are $V_- = \frac{4}{3}\pi(R - \sqrt{3}/2)^3$ and $V_+ = \frac{4}{3}\pi(R + \sqrt{3}/2)^3$, and the cubes are of unit volume. Therefore $V_- < N < V_+$. $V = \frac{4}{3}\pi R^3$ can be taken as the approximation sought. Rounding off we get the answer 33510321. To estimate the error of the upper bound:

$$\frac{V_+}{V} = \frac{\left(R + \frac{\sqrt{3}}{2}\right)^3}{R^3} = \left(1 + \frac{\sqrt{3}}{2R}\right)^3 = \left(1 + \frac{\sqrt{3}}{400}\right)^3 <$$
$$= 1 + \frac{3}{200} + \frac{3}{200^2} + \frac{1}{200^3} < 1 + \frac{1}{100}\left(\frac{3}{2} + \frac{3}{400} + \frac{1}{80000}\right) < 1.02.$$

   For the lower bound the estimation is similar.

7. Note that if $P_n(x)$ is a polynomial of even degree with a positive leading coefficient, then $\lim_{n\to\pm\infty} P_n(x) = +\infty$. Therefore it attains a minimum

at a point (by the theorem that a continuous functions in a closed interval attains its maximum and minimum values). Let x_0 be the point of absolute minimum of the polynomial. At this point $P_n'(x_0) = 0$ and $P_n''(x_0) \geq 0$. However, it is given that $P_n(x) \geq P_n''(x)$ for all x. Therefore, $P_n(x) \geq P_n(x_0) > P_n''(x_0) \geq 0$ for all x.

8. **Answer:** Ben will get \$10, Jen – \$1 , Glen will get nothing.

   We will analyze the game starting at the end.

   If one or two stones are left before a player's turn, he gets \$10. If 3 stones are left, he gets nothing. If a player is confronted with 4 stones, it is more profitable for him to remove one stone and get \$1 rather than remove 2 and get nothing. Hence a player who leaves 4 stones after his move, wins. Similarly, a player who leaves 7 stones after his move, finishes second (gets \$1), just like one who leaves 3. Hence in devising a winning strategy or a strategy for coming second (if there is no way to win), one can disregard 4 stones, as if they are not there. In other words, if the number of stones in the initial pile is reduced by 4 then the new pile leads to the same outcome. So the outcome of the game depends on the remainder modulo 4 of the number of stones in the original pile. If it is equal to 1 or 2, then it is Ben's win, 3 – Jen's, 4 – Glen's.

9. Multiply the right column (the hundreds) by 100, the middle column (the tens) by ten, and add them to the right column (the units). The value of the determinant will not change, while each entry of the right column will now be divisible by 17. Therefore the determinant is also divisible by 17.

10. **Answer:** Yes, this is possible.

    Mark a point on each of the the three edges emanating from a vertex A: the far end of the first edge, the midpoint of the second edge, the point of the third edge at the distance one-quarter of the edge length from A. Consider the plane through these points (it contains one vertex of the cube) and 7 planes parallel to it through the remaining vertices. To prove that these planes satisfy the requirements, consider the line through A that is orthogonal to the planes, and introduce coordinates on it by choosing A to be the origin and the distance from A to the constructed plane to be the unit of length. The lengths of the projections onto the line of the edge vectors v_1, v_2 and v_3, starting at A, will be 1, 2, and 4. Note that the vectors that connect A to the cube vertices are: 0, v_1, v_2, v_1+v_2, v_3, v_1+v_3, v_2+v_3, and $v_1+v_2+v_3$. It can be easily verified that the projections of the endpoints of these vectors onto the line are $0, 1, \ldots, 7$, so the planes are at equal distances from one another.

11. Set the unit of length to be the absolute value of the maximal vector of friction between the table and the crumb. Then $|F_i| > 1$ for $i = 1, 2, \ldots, N$. Denote the force applied to the crumb by the N ants be $\vec{S} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N$. It is given that $|\vec{S}| \leq 1$. We will now show that there is a vector $\vec{F}_j$, such that $|\vec{S} - \vec{F}_j| \geq |\vec{F}_j|$. The argument is by contradiction.

    Assume that $|\vec{S} - \vec{F}_j| < |\vec{F}_j|$ for all j. Then $|\vec{S} - \vec{F}_j|^2 < |\vec{F}_j|^2$, or $(\vec{S} - \vec{f}_j, \vec{S} - \vec{F}_j) < (\vec{F}_j, \vec{F}_j)$. Therefore, $\vec{S}^2 - 2(\vec{S}, \vec{F}_j) < 0$ for all j. Add up all these inequalities to get:

    $$0 > \sum_{j=1}^{N} (\vec{S}^2 - 2(\vec{S}, \vec{F}_j)) = N\vec{S}^2 - 2\left(\vec{S}, \sum_{j=1}^{N} \vec{F}_j\right) = $$
    $$= N\vec{S}^2 - 2(\vec{S}, \vec{S}) = (N-2)\vec{S}^2$$

    However, $N-2$ and $\vec{S}^2$ are non-negative. Contradiction.

12. **Answer:** 46 months.

    Solution. We will say that a distribution of salaries is *just* if every researcher is paid the same, and *almost just* if the highest salary exceeds the lowest one by not more than \$1.

    We start with showing that it can happen that a just distribution in not achievable in less than 46 months. Let the initial distribution be as follows: 13 researchers earn \$10 each, one researcher earns \$9, and one works for free. Then after n months this last researcher will be getting n dollars at most. If the salaries are all the same at this point, then the total payout does not exceed $15n$ dollars. On the other hand it is equal to $139 + 11n$ dollars, that is \$139 initially, increased by \$11 monthly. The inequality $139 + 11n \leq 15n$ then yields $n \geq 35$. On the other hand, if a distribution is just, then the total of $139 + 11n$ is divisible by 15. This happens if and only if $n \equiv 1 \pmod{15}$. Using the condition $n \geq 35$ we obtain for n the minimum possible value $n = 46$.

Next we will prove that an almost just distribution can be achieved from any initial distribution in 32 months or less, and that any almost just distribution can be upgraded to a just one in 14 months or less.

We will be following the rule: every month increase by \$1 the salaries of the 11 lowest-paid researchers (if several candidates are paid the same, any of them can be chosen).

We start with the second part – from almost justice to justice.
Clearly, if we follow our algorithm (increase the 11 lowest salaries monthly), then almost justice already established will never be broken. Note also that once in 15 consecutive months the total payout will be divisible by 15. (Among n consecutive terms of an arithmetic progression, whose difference is co-prime with n, there is a term divisible by n.) Note also that an almost just distribution with the total payout divisible by 15 is necessarily just. This proves that in 14 months or less one can achieve justice starting from almost justice.

Now the first part – how to achieve almost justice from any given distribution.

Note that a \$1 increase in all salaries does not make the current and all future distributions any more or less just, therefore an increase of \$1 in 11 salaries can be replaced with the \$1 reduction in the remaining four. We will use both operations in the sequel.

As already explained, every month we will increase the 11 lowest salaries. However, if this would lead to an increase in the salary of anyone already earning \$10, then instead of increasing the 11 lowest salaries we will reduce the four highest ones (clearly, if this situation arises, then the four highest earners get \$10 each). As we see, all along the way the highest salary will never exceed \$10, and the lowest one will never go down.

We will refer to those who earn \$9 or \$10 as *high earners*, and the rest – *low earners*. Note that once achieved, the status of high earner is never lost.

Now consider the two possible cases – no salary is ever reduced, and a reduction is made at least once.

Case A.
Assume that we have not reduced salaries at all (up to the establishment of almost justice). Each salary was increased 10 times at most, so the total monthly payout increased by \$150 at most. This took not more than 13 months (in 14 months the total payout would grow by $14 \cdot 11 = 154 > 150$ dollars). *End of Case A.*

Case B.
Assume that at least one reduction was required. (Recall that only a \$10 salary can be subject to a reduction.) We are going to prove that in 32 months or less everyone will be a high earner. For this end we will find bounds on the number of increases and reductions separately.

Starting with the increases, we will show by induction that *after the k-th increase (for $k \leq 9$) each researcher will be getting k dollars at least.* (That would mean in particular that not later than after 9 increases the distribution will become almost just, i.e., there may be 9 increases at most.) For $k = 0$ this is given. Assume the claim to be true for $k - 1$. We argue that it is true for k as well. Two cases will be dealt with separately: the first reduction came earlier than the k-th increase (Case α), and later than the k-th increase (Case β).

Case α. There has been one reduction at least before the k-th increase. As noticed earlier, prior to the reduction there are at least 4 high earners, and they remain in this capacity up to the k-th increase. But then the k-th increase has to include all of the low earners, as there are at most 11 of them. The induction claim follows. *End of Case α.*

Case β. Prior to the k-th increase there has not been a single reduction. Consider the group of researchers who earn $k - 1$ dollars after the increase number $k - 1$. We want to show that the k-th increase will include all of them. Assume the contrary – that the k-th increase does not affect at least one of them. That means that there are more than 11 people in the group. Note that if at some time in the future a member X of the group earns more than any given researcher Y (not necessarily from the group), the difference will not exceed \$1. (Indeed, the difference of more than \$1 would mean there had been an instance of X, but not Y, getting an increase, even though Y earned less than X at that point. That would contradict our algorithm.)

Consider the instance when the first salary reduction was required. At that time there were less that 11 researches with the salaries of less than \$10. Therefore there is someone in our group who earns \$10. Hence each of the 15 researchers earns 9 dollars at least – a just distribution has been achieved, and no reduction is required. This contradiction with the assumption of Case B proves the induction step. *End of Case* β.

By now we have proved the italicized statement and, as an immediate corollary, obtained an upper estimate of 9 for the number of salary increases.

Next we want to find an upper estimate for the number of reductions. Change the order of operations: make all the increases first (at most 9, as already shown), and the reductions afterwards – this will not affect the final result, even though at intermediate points some salaries may exceed \$10.

Consider the moment (under the new order of operations) when everyone has arrived at the level of \$9 at least. As shown above, this will happen after at most 9 increases. The total increase of the salaries above \$10 does not exceed $10 \cdot 9 = 90$ dollars, since each increase included at most 10 people whose upgraded salary was greater than \$10 (convince yourself, why all 11 subjects of the upgrade could not be of this kind), and there have been at most 9 such operations. With each reduction the total payout is decreased by \$4, hence after at most 23 reductions no salary will exceed \$10. But we have also learned that all of them are not less than \$9. So the distribution is now almost just, and this required at most $9 + 23 = 32$ months.

Grading criteria

We used the following symbols in grading:
+ (p) — a correct solution was given
+. (pt) — the solution contains some minor inaccuracies
± (pm) — the problem was generally solved but there were some flaws or gaps in the solution
+/2 (p2) — "half-way" progress was achieved
∓ (mp) — the problem has not been solved, but a correct approach to the solution was demonstrated
−. (mt) — the problem has not been solved, but there are some meaningful considerations
− (m) — the presented solution is incorrect
0 — no solution given to the problem.

A few characteristic cases are shown below, along with the grades given by the jury.

Problem 1.
± The correct formula for the concentration of a mixture is given. The requirement is written out explicitly in the form of an inequality, and the claim is made that concentrations and volumes can be found such that the inequality will be satisfied. However, no explicit counterexample is given.

Problem 3.
± The expression for the integral is written correctly but it is evaluated with an error.

Problem 4.
−. The limit is found only under the assumption that it exists.
∓ The solution is given for the particular case $m = 2$.
∓ The monotonicity is proved only in a particular case ($x^m > a$).

Problem 6.
+ If small terms are thrown away in the error evaluation without making

accurate estimates using inequalities, the grade is not reduced.
$\pm$ The correct count, but no error estimation.
$+/2$ The correct idea to use a sphere for counting but there is an error in its realization (e.g. an incorrect radius is used).
$\mp$ The idea of the evaluation of the volume of a ball is present.
$-$ The answer is given without justification, or it was obtained using a computer program.

Problem 7.
$\pm$ A correct argument applied to a point of minimum, but no proof given that such a point exists.

Problem 8.
$\pm$ The reasoning is correct but there is an error in the evaluation of the remainder of 2010 modulo 4, leading to a wrong answer.
$\mp$ The idea of analysis from the end or that of a cycle.

Problem 9.
$\pm$ A (correct) criterion of divisibility by 17 is used without proof.

Problem 10.
$\mp$ It is shown how to construct 4 of the required planes through the vertices of a face of the cube, with further reference to continuity considerations.

Problem 12.
$\mp$ The correct numerical answer and the distribution of salaries that requires this number of months are given.
$\mp$ Proved that increasing the salaries of the lowest-paid researchers is an optimal strategy.