1. Calculate the sum of the series \( \sum_{n=0}^{\infty} \frac{1}{n^2 + 4n + 3} \).

Answer: \( \frac{3}{4} \).

The answer in decimal form (for the Blitz): 0.75.

Solution.

\[
\sum_{n=0}^{\infty} \frac{1}{n^2 + 4n + 3} = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+3)} = \sum_{n=0}^{\infty} \frac{1}{2} \cdot \frac{(n+3) - (n+1)}{(n+1)(n+3)} = \\
\frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+3} \right) = \frac{1}{2} \cdot \lim_{m \to \infty} \sum_{n=0}^{m} \left( \frac{1}{n+1} - \frac{1}{n+3} \right) = \\
\frac{1}{2} \cdot \lim_{m \to \infty} \left( \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{m} - \frac{1}{m+2} + \frac{1}{m+1} - \frac{1}{m+3} \right) = \\
\frac{1}{2} \cdot \lim_{m \to \infty} \left( \frac{1}{2} - \frac{1}{m+2} - \frac{1}{m+3} \right) = \frac{1}{2} \left( 1 + \frac{1}{2} \right) = \frac{3}{4}.
\]

2. Find the first digit of the number \( 2^{10000000000} = 2^{10^{10}} \) written in decimal form using your calculator.

Answer: 4.

Solution. Let us use the formula \( a^x = 10^{x \lg a} \) for \( a = 2, \ x = 10^{10} \).

Thus we have

\[
\lg 2 = 0.30102999566398 \ldots, \quad 10^{10} \lg 2 = 3010299956, 6398 \ldots;
\]

\[
2^{10^{10}} = 10^{10^{10} \lg 2} = 10^{3010299956.6398 \ldots} = \\
10^{0.6398 \ldots} \cdot 10^{3010299956} = 4.363 \ldots \cdot 10^{3010299956} = 4363 \ldots
\]

3. Calculate the integral

\[
\int_0^\pi \frac{dx}{2 + \cos^2 x}.
\]

(In the Blitz participants are allowed to use calculators during the last stage, in order to calculate the approximate numerical value).
Answer: \(\frac{\pi}{\sqrt{6}}\).

The answer in decimal form (for the Blitz): 1, 28.

Solution. Let us first find the antiderivative of the integrand. We can use the standard substitution \(u = \tan x\), \(du = \frac{dx}{\cos^2 x}\):

\[
\int \frac{dx}{2 + \cos^2 x} = \int \frac{du}{2u^2 + 3} = \frac{1}{\sqrt{6}} \arctan \left( \sqrt{\frac{3}{2}} u \right) + C.
\]

Note that when the variable \(u\) changes from \(-\infty\) to \(+\infty\), \(x\) changes from \(-\pi/2\) to \(\pi/2\), and the limits of integration in the given problem go beyond this interval. Therefore, we cannot utilize the Newton-Leibniz formula directly. However, we can use the fact that the integrand is symmetric about the axis \(x = \pi/2\) and therefore

\[
\int_0^\pi \frac{dx}{2 + \cos^2 x} = 2 \int_0^{\pi/2} \frac{dx}{2 + \cos^2 x} = 2 \int_0^{+\infty} \frac{du}{2u^2 + 3} = 2 \left( \frac{1}{\sqrt{6}} \arctan \left( \sqrt{\frac{3}{2}} u \right) \right) \bigg|_0^{+\infty} = \frac{\pi}{\sqrt{6}}.
\]

4. What is the greatest weight, expressed in whole grams, that cannot be obtained using only 6, 10 and 15 gram weights?

Answer: 29 grams.

Solution. First, let us prove that 29 grams cannot be obtained using only 6, 10 and 15 gram weights. The number 29 is not divisible by 5, and therefore, it is clear that we could not obtain it using only 10 and 15 gram weights. Thus, we would have to use at least one 6 gram weight. Similarly, since 29 is not divisible by 3, we would have to use at least one 10 gram weight, and since it is not divisible by 2 we would have to use at least one 15 gram weight. But the smallest weight that can be obtained using at least one 6 gram weight, at least one 10 gram weight and at least one 15 gram weight is 31.

Now, let us prove that any weight greater than 29 grams can be obtained using these weights only. Let us check the weights 30-35 grams
directly:

\[
\begin{align*}
30 &= 15 + 15, \\
31 &= 15 + 10 + 6, \\
32 &= 10 + 10 + 6 + 6, \\
33 &= 15 + 6 + 6 + 6, \\
34 &= 10 + 6 + 6 + 6 + 6, \\
35 &= 15 + 10 + 10.
\end{align*}
\]

Any weight greater than 35 can be obtained by adding one or more 6 gram weights to one of these six weights.

5. We have a set of 2010 integers, none of which is divisible by any other. Let us call an integer \textit{good}, if the product of two other integers is divisible by its square. What is the maximal possible number of ”good” integers?

**Answer:** 2008.

**Solution.**

Let’s give an example of a set containing 2008 ”good” integers. Let \(x_k = 2^k \cdot 3^{2009-k} (k = 0, 1, \ldots, 2009)\).

This is a geometric sequence with ratio 3/2. The square of each of its elements, except for the first and the last, is equal to the product of the two adjacent elements (the previous and the following). Therefore, all of these elements are \textit{good}.

Let us prove that any set of 2010 integers satisfying the previously mentioned condition contains at least 2 ”bad” (i.e. not ”good”) numbers. The maximal number of any set is always a ”bad” number, because its square is greater than the product of any other two numbers. Let us denote the maximal number of the set as \(x\).

For any non-maximal number \(z\), a prime number \(p\), which is divisible by a greater power of \(p\) than \(x\), can be found. Otherwise, \(x\) would be divisible by \(z\). Let us examine the maximal power \(k\), for which there are numbers in the set, that are divisible by \(p^k\). Let us denote the maximal number, which is divisible by \(p^k\), as \(y\), and prove that \(y\) is a
"bad" number. In order to do this, let us examine the product of any two numbers (other than $y$) that belong to the set. If at least one of them is not divisible by $p^k$, then their product is divisible by a lesser power of $p$ than $y^2$. If on the other hand they are both divisible by $p^k$, then their product has to be less than $y^2$. In both cases the product cannot be divisible by $y^2$, and therefore $y$ is a bad number.

6. Suppose that we have a tetrahedron. We call a plane balanced if all the vertices of the tetrahedron are equidistant from it. How many balanced planes are there?

**Answer:** 7.

**Solution.** Let us note, that in order for any given plane to be balanced, all the vertices of the tetrahedron must be located in two planes that are equidistant from the given plane, and these two planes must lie on different sides of the given plane. (All the vertices cannot be located on the same side of the given plane, because in this case they would all have to lie on the same plane). If three of the vertices lie on one side of the balanced plane, and the fourth vertex lies on the other side, then the plane is parallel to one of the tetrahedron’s bases and bisects the altitude to this base. The number of such planes is equal to the number of a tetrahedron’s facets, i.e., 4. If two of the vertices lie on one side of the balanced plane, then the plane is parallel to two intersecting edges of the tetrahedron and equidistant from two parallel planes which intersect these edges. The number of such planes is equal to the number of a tetrahedron’s pairs of intersecting edges, i.e., 3. Therefore, there are $4 + 3 = 7$ balanced planes.

7. Jack and Jill are playing a game similar to Tic-Tac-Toe. The rules of the game are as follows. There is an unlimited number of $3 \times 3$ grids. Jack and Jill take turns, but Jill always goes first. During each of her turns Jill makes two X marks. She can either place both on the same grid, or place them on different grids. During his turn Jack can make one O mark on any grid he chooses. Jill’s goal is to fill all of the squares of at least one of the grids with Xs. If both Jack and Jill use optimal strategies, what is the minimal number of turns in which she is guaranteed to achieve this goal?

**Answer:** 128.
Solution. First, let us describe Jill’s optimal strategy. In order to win using the minimal number of moves, Jill must start by marking two empty boards (placing an X on each of them) during each of her turns, until there are 64 grids with at least one X and no Os on them. Since after every move that she makes the number of grids with at least one X and no Os increases by two, and after every move that Jack makes their number decreases by no more than 1, she will accomplish this in 64 moves at most. If after this Jill starts placing an additional X on two of these 64 grids during each of her turns, she will have 32 grids with two Xs and no Os after she makes 32 more moves. Thus, if she continues using this strategy, after \(64 + 32 + 16 + 8 + 4 + 2 + 1 = 127\) moves she will have one grid with 7 Xs and no Os on it. She will then have to make one more move (place two more Xs on this grid) in order to win.

Now, let us examine Jack’s optimal strategy. In order to stay in the game for as long as possible, during each of his turns he must place an O on a grid with the greatest number of Xs, thus ”spoiling” this grid. Let us prove that this is indeed his optimal strategy.

Let us assign a number to each grid. If the grid is ”spoiled”, its number will be zero. Otherwise, the grid’s number will be \(2^k - 1\) where \(k\) is the number of Xs on the grid. Note that according to this rule the number of a grid on which there are no Xs is also zero. Let us examine how the sum of the numbers of all of the grids changes during the game. At the beginning this sum equals zero. In order for Jill to win, the sum must be equal to or greater than 127. If Jill places Xs on two ”unspoiled” grids, she increases the sum by \(2^m + 2^n\), where \(m\) is the number of Xs that she had previously placed on the first grid, and \(n\) - the number of Xs that she had previously placed on the second. Jack, during his turn, places an O on a grid with \(k\) Xs, thus decreasing the sum by \(2^k - 1\).

Since \(m \leq k - 1\) \(n \leq k - 1\) after both Jill and Jack make their moves, the sum will increase by no more than \((2^{k-1} + 2^{k-1}) - (2^k - 1) = 1\). If Jill places one of her Xs on a ”spoiled” grid, or places two Xs on the same grid, the sum does not increase at all. Thus, after each turn the sum increases by no more than 1, and therefore Jack will be able to stay in the game for at least 127 moves.

8. How many different matrices of size 2 × 2 satisfy the following condi-
tions:

a) all the four elements of which equal 1, 0 or $-1$

b) raising the matrix to the

power of $2010!$ produces an identity matrix?

Answer: 24.

Solution. The characteristic polynomial of the desired matrix $A$ has the form

$$x^2 - (\text{tr} \ A)x + \det A = 0,$$

where $\text{tr} \ A$ and $\det A$ denote the trace and the determinant of matrix $A$, respectively. Since one of the powers of matrix $A$ is an identity matrix $\det(A) = \pm 1$ and the eigenvalues of the matrix are complex roots of 1. Conversely, if the roots of the characteristic polynomial are roots of 1, of a power which is a divisor of the number $2010!$, and the matrix is diagonalizable, then it satisfies the conditions.

In addition $|\text{tr}(A)| \leq 2$ since the matrix elements according to the condition (a) do not exceed 1. Let us examine all the possible values of the trace and the determinant separately:

Case 1. $|\text{tr} \ A| = 2$. In this case both of the matrix’s eigenvalues equal either 1 or $-1$. Since a certain power of the matrix is equal to the identity matrix $E$, it is diagonalizable, and therefore, $A = \pm E$. In this case we have two matrices that satisfy the conditions of the problem.

Case 2. $\text{tr} \ A = 0$, $\det A = -1$. The matrix’s $A$ eigenvalues equal either 1 or $-1$. On the main diagonals are either zeros, or 1 and $-1$. In the former case, on the secondary diagonal we have either two 1s or two $-1$s. Thus, we have two matrices. In the latter case, the product of the numbers on the secondary diagonal is zero. There are two possible ways of placing numbers on the main diagonal, and 5 possible ways of placing numbers on the secondary diagonal. Thus we have ten more matrices, and a total of 12 matrices.

Case 3. $\text{tr} \ A = 0$, $\det A = 1$. The matrix’s $A$ eigenvalues equal either $i$ or $-i$. If on the main diagonal we have 1 and $-1$, then the product of the numbers on the secondary diagonal must equal $-2$, which is impossible. If on the main diagonal we have zeros, then the numbers on the secondary diagonal must be 1 and $-1$. In this case there are two matrices.
Case 4. \( \text{tr } A = -1 \). In this case the characteristic polynomial has the form \( x^2 + x + \det A = 0 \). If \( \det A = -1 \), then the roots of this polynomial are not roots of 1, and therefore, this is not the desired matrix. If \( \det A = 1 \), then the roots of this polynomial are cube roots of 1. In this case, on the main diagonal we have 1 and 0 (two possibilities), and on the secondary diagonal we have either two 1s, or two \(-1\)s. Thus, there are 4 matrices.

Case 5. Since matrix \(-A\) satisfies the same conditions as matrix \(A\), in this case we have the same number of matrices, i.e. 4.

Thus, we have a total of \( 2 + 12 + 2 + 4 + 4 = 24 \) matrices.